Non Perturbative Renormalization Group, momentum dependence of n-point functions and the transition temperature of the weakly interacting Bose gas

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(Dated: February 2, 2008)

We propose a new approximation scheme to solve the Non Perturbative Renormalization Group equations and obtain the full momentum dependence of n-point functions. This scheme involves an iteration procedure built on an extension of the Local Potential Approximation commonly used within the Non Perturbative Renormalization Group. Perturbative and scaling regimes are accurately reproduced. The method is applied to the calculation of the shift ΔT_c in the transition temperature of the weakly repulsive Bose gas, a quantity which is very sensitive to all momenta intermediate between these two regions. The leading order result is in agreement with lattice calculations, albeit with a theoretical uncertainty of about 25%. The next-to-leading order differs by about 10% from the best accepted result.

PACS numbers: 03.75.Fi,05.30.Jp

In this paper we present a new and systematic method of approximation for the Non Perturbative Renormalisation Group (NPRG)[1, 2, 3], which allows one to get, in a simple way, the full momentum dependence of the n-point functions. During the last years, the NPRG has been applied successfully to a variety of physical problems (for reviews, see e.g. [4, 5]). In most cases, the solution of the NPRG equations involves a derivative expansion which only allows for the determination of the n-point functions and their derivatives at vanishing external momenta. There are however situations where the knowledge of the full momentum dependence of the n-point functions is needed. A simple example is that of the calculation of the shift ΔT_c of the transition temperature of a weakly interacting Bose gas [6]. This will be used here as a test of our new approximation scheme.

In order to present our method, we consider a scalar ϕ^4 theory in d dimension with $O(N)$ symmetry:

$$\mathcal{S} = \int \left\{ \frac{1}{2} \left[\nabla\phi(x)\right]^2 + \frac{1}{2} r\phi^2(x) + \frac{u}{4!} \left[\phi^2(x)\right]^2 \right\} \mathrm{d}^d x \,. \quad (1)$$

The field $\phi(x)$ has N real components $\phi_i(x)$, with $i = 1, \cdots, N$.

The starting point of the NPRG is a modification of the classical action (1), to which is added

$$\Delta S_\kappa[\phi] = \frac{1}{2} \int \frac{\mathrm{d}^d p}{(2\pi)^d} R_\kappa(p) \phi_i(p) \phi_i(-p). \quad (2)$$

The role of ΔS_κ is to suppress the fluctuations with momenta $p \lesssim \kappa$, while leaving unaffected the modes with $p \gtrsim \kappa$. Thus, typically $R_\kappa(p) \to \kappa^2$ when $p \ll \kappa$, and $R_\kappa(p) \to 0$ when $p \gtrsim \kappa$. There is a large freedom in the choice of $R_\kappa(p)$, abundantly discussed in the literature [7, 8, 9, 10]. We have used a cut-off function proposed by Litim [10]: $R_\kappa(p) \propto (\kappa^2 - p^2)\, \theta(\kappa^2 - p^2)$, which allows many calculations to be done analytically.

The NPRG equations can be written as an infinite hierarchy of flow equations describing how the $n-$point functions evolve with the infrared cut-off κ [1, 2, 3]. Here, we write explicitly only the equation for the self-energy Σ and that for the 4-point function $\Gamma^{(4)}$, which is enough for the purpose of explaining our approximation scheme. For vanishing fields, $\langle\phi\rangle = 0$, the flow equation for $\Sigma_{ij}(\kappa; p) \equiv \delta_{ij}\Sigma(\kappa; p)$ reads

$$\delta_{ij}\partial_\kappa \Sigma(\kappa; p) = -\frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \partial_\kappa R_\kappa(q) G_\kappa^2(q) \Gamma^{(4)}_{ijll}(\kappa; p, -p, q, -q), \quad (3)$$

and that for $\Gamma^{(4)}_{ijkl}(\kappa; p_1, p_2, p_3, p_4)$ is given by

$$\begin{aligned} \partial_\kappa \Gamma^{(4)}_{ijkl}(\kappa; p_1, p_2, p_3, p_4) = \int \frac{d^d q}{(2\pi)^d} \partial_\kappa R_\kappa(q) G^2(\kappa; q) \\ \times \Big\{ \Gamma^{(4)}_{ijmn}(\kappa; p_1, p_2, q, -q - p_1 - p_2) G(\kappa; q + p_1 + p_2) \\ \times \Gamma^{(4)}_{klnm}(\kappa; p_3, p_4, q - p_3 - p_4, -q) + \text{permutations} \\ - \frac{1}{2}\, \Gamma^{(6)}_{ijklmm}(\kappa; p_1, p_2, p_3, p_4, q, -q) \Big\}. \end{aligned} \quad (4)$$

More generally, the equations for $\Gamma^{(n)}$ involves all the $\Gamma^{(m)}$ with $m \le n \le n+2$. In these equations, $G^{-1}(\kappa; p) = p^2 + R_\kappa(p) + \Sigma(\kappa, p)$. These equations are to be solved with the boundary condition that as $\kappa \to \Lambda$, with Λ the microscopic scale, the n-point functions take their classical values, read on the action (1). In particular,

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$\Gamma^{(4)}_{ijkl}(\Lambda; p_1, p_2, p_3, p_4) = (u/3)(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. As $\kappa \to 0$ the $n-$point functions go to their physical values.

Clearly, in general, the NPRG hierarchy of equations can only be solved approximately. But a virtue of the NPRG is precisely to suggest approximations which are not easily formulated in other, more conventional, approaches. A popular one is the derivative expansion [1, 11]. In lowest order it consists in ignoring the momentum dependence of vertices in the right hand side of the flow equation, as well as field renormalization. In this approximation, usually referred to as the Local Potential Approximation (LPA), the hierarchy collapses into a closed equation for the effective potential $V_\kappa(\phi^2)$. An interesting improvement of the LPA, which we refer to as the LPA', takes into account a running field renormalisation constant Z_κ and allows for a non trivial anomalous dimension, determined from the cut-off dependence of Z_κ, $\eta = -\kappa\partial_\kappa \ln Z_\kappa$ [1]. The solution of the LPA' is well documented in the literature (see e.g. [5, 9]). It will be used as an input in our approximation scheme. Let us emphasize that for massless theories, the derivative expansion is an expansion in terms of p^2/κ^2 and therefore, in the physical limit $\kappa \to 0$, it makes sense only for vanishing external momenta, $p = 0$. Higher orders in the derivative expansion [9] may improve the accuracy of the LPA but, again, this concerns only the n-point functions or their derivatives at zero momenta.

In order to get the full momentum dependence of the n-point functions, we propose here to solve the hierarchy of the NPRG equations through an iteration procedure. The iteration of the NPRG equations starting with the classical values of the n-point functions as initial input reconstructs the usual loop expansion — see e.g. [12, 13]. Thus one expects generically the iterations of the exact flow equations to correctly account for the high momentum behaviour. This is not so however for the low momentum region. There, our approximation scheme will exploit the fact that the LPA' is a good approximation in the limit of vanishing momenta.

The iteration procedure starts with an initial guess for the n-point functions to be used in the right hand side of the flow equations. Integrating the flow equation of a given n-point function gives then the *leading order* (LO) estimate for that n-point function. Inserting the leading order of the n-point functions thus obtained in the right hand side of the flow equations and integrating gives then the *next-to-leading order* (NLO) estimate of the n-point functions. And so on.

There is no small parameter controling the convergence of the process, and the terminology LO, NLO, etc. that we just used refers merely to the number of iterations involved in the calculation of the n-point function considered. Obviously, the calculations become increasingly complicated as the number of iterations increases, and it is essential that the initial guess be as close as possible to the exact solution so that only one or two iterations suffice to get an accurate result. The main effort focuses then on the construction of such a good initial guess for the solution, to which we now turn.

We shall be guided by a crucial property of the NPRG, that of decoupling of the various momentum scales: for given external momenta $p \lesssim \kappa$, the flow of the n-point functions is dominated by internal momenta $q \lesssim \kappa$ (the derivative $\partial_\kappa R_\kappa(q)$ limits the range of integration in the flow equations to $q \lesssim \kappa$), and when κ becomes lower than p the flow essentially stops (the external momenta playing the role of infrared regulators).

Consider first the 2-point function. A reasonable initial guess for the propagator to be used in the r.h.s. of the flow equations when calculating the leading order estimate of the n-point functions is the LPA' propagator $G^{-1}_{LPA'}(\kappa; p) = Z_\kappa p^2 + m^2_\kappa + R_\kappa(p)$, where m^2_κ is the running LPA' mass: it is a good approximation when $p \lesssim \kappa$, and for $p \gtrsim \kappa$, $G^{-1}(\kappa; p)$ goes quickly to zero and its precise form does not matter.

A good initial ansatz for the other n-point functions (with $n > 2$) will be obtained by solving the flow equations for the $\Gamma^{(n)}$'s, with the following three approximations.

Our first approximation (A1) assumes that for $q \lesssim \kappa$, and any set of external momenta $\{p_1, p_2, ..., p_n\}$ we have

$$\left| \frac{\Gamma^{(n)}(\kappa; p_1, ..., p_{n-1} + q, p_n - q) - \Gamma^{(n)}(\kappa; p_1, ..., p_n)}{\Gamma^{(n)}(\kappa; p_1, ..., p_n)} \right| . \ll 1 \tag{5}$$

This is certainly true for momenta $\{p_1, p_2, ..., p_n\}$ whose norms are much larger than κ, if $\Gamma^{(n)}$ is a smooth function of its arguments. Similarly, for vanishing momenta, and assuming again that $\Gamma^{(n)}$ is a smooth function, the condition above is equivalent to saying that one can expand in powers of q^2/κ^2, which leads in zeroth order to the LPA, a good approximation. Based on this assumption, we shall set $q = 0$ in all $\Gamma^{(n)}$ $(n > 2)$, and factor them out of the integral in the r.h.s. of the flow equations.

The second approximation (A2) concerns the propagators in the flow equation, for which we make the replacements:

$$G(p + q) \longrightarrow G_{LPA'}(q)\, \Theta(1 - \frac{\alpha^2 p^2}{\kappa^2}) \tag{6}$$

where α is an adjustable parameter. As an illustration of the quality of this approximation, we show in Fig. 1 the ratio $I_\kappa(p)/I_\kappa(0)$ where

$$I_\kappa(p) \equiv \int \frac{d^d q}{(2\pi)^d} \partial_\kappa R_\kappa(q) G^2(\kappa; q) G(\kappa; p + q) \tag{7}$$

is the integral wich remains in Eq. (4) after approximation A1. This ratio, as a function of p^2/κ^2, looks indeed like a step function, with a weak residual κ dependence. The approximation A2 amounts to truncate severely the high momentum tails of the propagators. This causes a slight inaccuracy at high momenta, and a dependence of the leading order results on the value of α. One may fix α so that the inflexion point of the curve in Fig. 1 is at $\alpha p = \kappa$. One then obtains typically $\alpha \approx .9$. One can

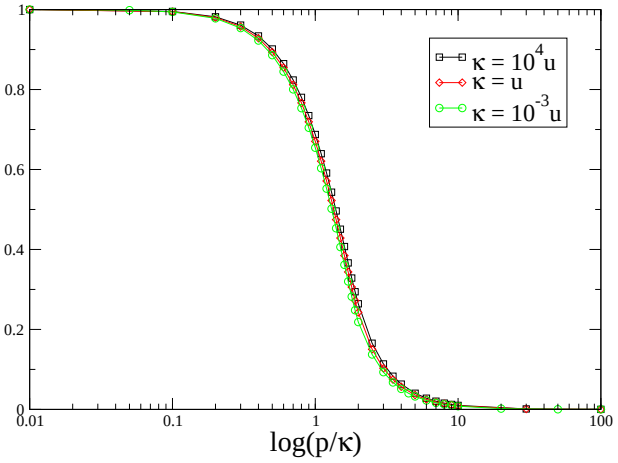

FIG. 1: The function $I_\kappa(p)/I_\kappa(0)$ as a function of $\ln(p/\kappa)$ calculated with the leading order propagator at criticality

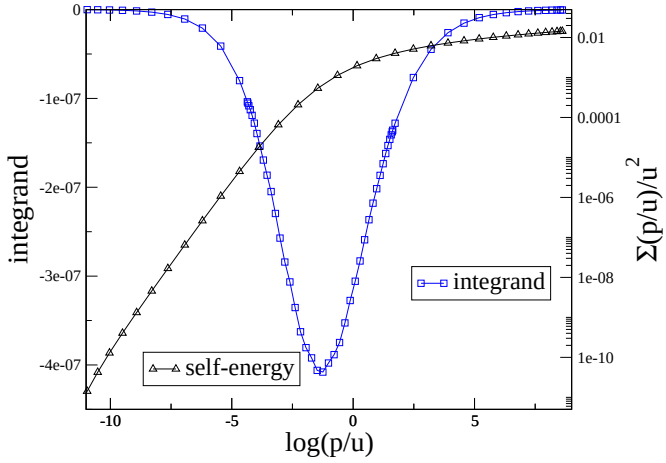

FIG. 2: The leading order function $\sigma(x = p/u)$ at criticality, and the integrand of Eq. 10 as a function of $\ln(p/u)$, for $u \lesssim 10^{-4}$ for which $\sigma(x)$ is independent of u [16]

also adjust α so that the integral over κ of $I_\kappa(p)$ is identical to that of $I_\kappa(0)\Theta(\kappa^2 - \alpha^2 p^2)$. This yields typically $\alpha \approx .6$. We regard these two possible choices as extremes and adopt the value $\alpha = .75 \pm .15$ for our leading order estimate.

The third approximation (A3) concerns the function $\Gamma^{(n+2)}$ in the equation for $\Gamma^{(n)}$. For this we use an ansatz inspired by the expressions of the various n-point functions in the large N limit [14]. To be specific, consider $\Gamma^{(6)}$ in the equation for $\Gamma^{(4)}$. The approximation A3, combined with A1 and A2, leads to the result that the contribution of $\Gamma^{(6)}$ to the r.h.s. of Eq. (4) is proportional to that of the other terms, the proportionality coefficient F_κ being only a function of κ. The same proportionality also holds in the LPA regime, which allows us to use the LPA to determine F_κ.

Approximations A1-A3, when applied to Eq. (4), yield the following equation for our initial guess for $\Gamma^{(4)}$:

$$\partial_\kappa \Gamma^{(4)}_{ijkl}(\kappa; p_1, p_2, p_3, p_4) = I^{(3)}_\kappa(0)\,(1 - F_\kappa) \times \Big\{ \Theta\big(\kappa^2 - \alpha^2 (p_1 + p_2)^2\big)\big)\, \Gamma^{(4)}_{ijmn}(\kappa; p_1, p_2, 0, -p_1 - p_2) \times \Gamma^{(4)}_{klnm}(\kappa; p_3, p_4, -p_3 - p_4, 0) + \text{permutations} \Big\}. \tag{8}$$

This equation can be solved analytically in terms of the solution of the LPA'[14]. This is done by steps, starting form the momentum domain $\alpha^2(p_1 + p_2)^2, \alpha^2(p_1 + p_3)^2, \alpha^2(p_1 + p_4)^2 \leq \kappa^2$, where it can be verified that the solution is that of the LPA' itself. The solution can be written explicitely in the various momentum regions defined by the Θ-functions occuring in Eq. (8).

The leading order result for the self-energy is obtained by using the solution of Eq. (8) together with $G_{LPA'}$ in the r.h.s. of Eq. (3) and integrating numerically over κ. The resulting self-energy at criticality, i.e., for the value of r that yields a vanishing physical mass, is shown in Fig. 2 for $N = 2$ and $d = 3$. It has good behaviour at both low and high momenta, independently of the value of α: In the scaling regime, one recovers the expected result $p^2 + \Sigma(p) \propto p^{2-\eta}$, where the anomalous dimension is $\eta \simeq 0.043$; this is only slightly larger than values determined by the most accurate available methods [15]. One can show [14] that the value of η coincides with that obtained within the LPA', but now it results directly from the momentum dependence of the self-energy, rather than the cut-off dependence of the field renormalization. At large momenta, the logarithmic behaviour given by perturbation theory is reproduced. However, the coefficient of the logarithm differs by about 10% from the expected one. This problem, cured in NLO, finds its origin in the approximation A2 which truncates the high momentum tails of the propagators.

We have applied this approximation scheme to the calculation of the shift ΔT_c of the transition temperature of a weakly interacting Bose gas. It has been shown recently that ΔT_c is linear in $an^{1/3}$ [16], where a is the scattering length and n the particle density:

$$\frac{\Delta T_c}{T_c^0} = c\; a n^{1/3}. \tag{9}$$

Here T_c^0 is the condensation temperature of the ideal gas, given by $n\lambda_c^3 = \zeta(3/2)$ with $\lambda_c^2 = 2\pi/mT_c^0$ (m is the boson mass), and $\Delta T_c = T_c - T_c^0$ with T_c the transition temperature of the interacting system. As shown in Ref. [16], the coefficient c can be related to the change $\Delta\langle\phi^2\rangle$ of the fluctuation of the field described by the action (1): $c = -256\pi^3 \left(\zeta(3/2)\right)^{-4/3} (\Delta\langle\phi^2\rangle/Nu)$, with $N = 2$. The parameters r and u in (1) are then related

to the scattering length a and the chemical potential μ by: $u = 96\pi^2 a/\lambda^2$, and $r = -2mT\mu$ [19].

The best numerical estimates for $\Delta\langle\phi^2\rangle$, and hence for c, are those which have been obtained using the lattice technique by two groups, with the results: $c = 1.32 \pm 0.02$ [17] and $c = 1.29 \pm 0.05$ [18]. The availabilty of these results has turned the calculation of c into a testing ground for other non perturbative methods: expansion in $1/N$ [19, 20], optimized perturbation theory [21], resummed perturbative calculations to high loop orders [22]. Note that while the latter methods yield critical exponents with several significant digits, they predict c with only a 10% accuracy. This illustrates the difficulty of getting an accurate determination of c using (semi) analytical techniques.

To understand better the origin of the difficulty, let us write $\Delta\langle\phi^2\rangle$ as the following integral [16]

$$\frac{\Delta\langle\phi_i^2\rangle}{Nu} = -\int \frac{dx}{2\pi^2}\,\frac{\sigma(x)}{x^2+\sigma(x)}, \qquad (10)$$

where $\sigma(x) = u^{-2}\Sigma(p = xu)$, with $\Sigma(p)$ the self-energy at criticality, i.e., $\Sigma(0) = 0$. The integrand of Eq. (10), at leading order, is shown in Fig. 2. The momentum at the minimum defines the typical scale which separates the scaling region from the high momentum region where perturbation theory applies. The difficulty in getting a precise evaluation of the integral (10) is that it requires an accurate determination of $\Sigma(p)$ in a large region of momenta including the crossover region between two different physical regimes [6, 19].

In leading order, we obtain $c \approx 1.3 \pm .3$, in agreement with lattice results [23]. The large uncertainty comes from the freedom in the choice of α that we have discussed above. We have also studied the NLO [14]. This involves the leading order for $\Gamma^{(4)}$, which requires the initial guess for $\Gamma^{(6)}$. The latter is obtained by following the same strategy as for $\Gamma^{(4)}$. The NLO result for c still depends on α, however in such a way that one can fix α from a criterium of fastest apparent convergence: in fact there is a value of α for which the correction to the LO result vanishes. This value is $\alpha = .83$ leading to $c = 1.44$.

Results also exist for other values of N, from lattice calcultions [17, 18] or perturbation theory to 7-loops [22]. For $N = 1$, we get $c = 1.20$, to be compared with $c = 1.09 \pm 0.09$ (lattice) or 1.07 ± 0.10 (7-loops); for $N = 3$, we get $c = 1.62$, to be compared with 1.43 ± 0.11 (7-loops); for $N = 4$, we get $c = 1.79$, to be compared with $1.60 \pm .10$ (lattice) and 1.54 ± 0.11 (7-loops). Our results thus differ by 10 to 15% from the best estimates. In its present numerical implementation, our method looses accuracy in the large N limit, where we get $c \approx 2.8$ instead of the exact result $c = 2.3$ [19].

In summary, we have proposed a simple method to get the momentum dependence of the n-point functions using the NPRG. Using well motivated approximations we have produced a simple solution which provides an excellent starting point for a more accurate iterative solution, as revealed by its application to the calculation of ΔT_c. Although there is no small parameter that controls the convergence of the iterative procedure, the NLO calculation suggests that the scheme is stable.

[1] C.Wetterich, Phys. Lett., **B301**, 90 (1993).
[2] U.Ellwanger, Z.Phys., **C58**, 619 (1993).
[3] T.R.Morris, Int. J. Mod. Phys., **A9**, 2411 (1994).
[4] C. Bagnuls and C. Bervillier, Phys. Rept. **348**, 91 (2001).
[5] J.Berges, N.Tetradis, C.Wetterich, Phys. Rept., **363**, 223 (2002).
[6] G. Baym, J.-P. Blaizot, M. Holzmann, F. Laloë, and D. Vautherin, Eur. Phys. J. **B24**, 107 (2001).
[7] R.D.Ball, P.E.Haagensen, J.I.Latorre and E. Moreno, Phys. Lett., **B347**, 80 (1995).
[8] J.Comellas, Nucl. Phys., **B509**, 662 (1998).
[9] L.Canet, B.Delamotte, D.Mouhanna and J.Vidal, Phys. Rev. **D67** 065004 (2003).
[10] D.Litim, Phys. Lett., **B486**, 92 (2000); Phys. Rev., **D64**, 105007 (2001); Nucl. Phys., **B631**, 128 (2002); Int.J.Mod.Phys., **A16**, 2081 (2001).
[11] T.R.Morris, Phys. Lett. **B329**, 241 (1994).
[12] J. Polchinski, Nucl. Phys. **B231**, 269 (1984).
[13] M. Bonini, M. D'Attanasio, G. Marchesini, Nucl.Phys. **B409**, 441 (1993).
[14] J.-P. Blaizot, R.Méndez Galain, N. Wschebor, in preparation.
[15] R. Guida and J. Zinn-justin, J. Phys A**31**, 8103 (1998).
[16] G. Baym, J.-P. Blaizot, M. Holzmann, F. Laloë, and D. Vautherin, Phys. Rev. Lett. **83**, 1703 (1999).
[17] P. Arnold and G. Moore, Phys. Rev. Lett. **87**, 120401 (2001).
[18] V.A. Kashurnikov, N. V. Prokof'ev, and B. V. Svistunov, Phys. Rev. Lett. **87**, 120402 (2001).
[19] G. Baym, J-P. Blaizot and J. Zinn-Justin, Europhys. Lett. **49**, 150 (2000).
[20] P. Arnold and B. Tomasik, Phys. Rev. A **62**, 063604 (2000).
[21] F. de Souza Cruz, M.B. Pinto, and R.O. Ramos, Phys. Rev. **B64**, 014515 (2001); Phys. Rev. A **65**, 053613 (2002).
[22] B. M. Kastening, Phys. Rev. A **69**, 043613 (2004).
[23] S.Ledowski, N. Hasslmann and P. Kopietz, Phys. Rev. **A 69**, 061601(R) (2004) have recently used the NPRG to calculate the coefficient c, and obtained $c = 1.23$. A critical discussion of their approach will be presented in [14]. We only observe here that in solving the equation for the 4-point function they ignore terms that we find important.